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TWO-PERIOD, STOCHASTIC SUPPLY-CHAIN MODELS WITH RECOURSE FOR NAVAL SURFACE WARFARE

by

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March 2004

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We model the minimum-cost procurement and allocation of anti-ship cruise missiles to naval combat ships as a two-period stochastic integer program. Discrete scenarios in two periods define "demands" for missiles (i.e., targets and number of missiles required to kill those targets), which must be met with sufficiently high probabilities. After the former combat period, ships may replenish their inventories from a depot if desired and if the available depot inventory suffices. A force commander optimizes ship-to-target assignments to meet demands. The basic model solves slowly, so we add constraints to enforce reasonable operational directives, and add valid inequalities. These improvements reduce the solution time by 95% for the test case. Instances with up to six ships and five scenarios in each period then solve in less than one hour on a 2 GHz personal computer.

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TWO-PERIOD, STOCHASTIC SUPPLY-CHAIN MODELS WITH RECOURSE FOR NAVAL SURFACE WARFARE

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EXECUTIVE SUMMARY

Procuring and allocating anti-ship cruise missiles to combat ships is a difficult problem facing modern navies, which must plan for many possible combat scenarios. The number of missiles required by each ship over an entire conflict may exceed its capacity, so a ship may need to return to a depot at a port to load more missiles between successive periods of combat. We consider two periods of combat, and seek to determine minimum-cost initial ship load-outs plus depot-level inventory, while ensuring, with sufficiently high probability, that all ships can satisfy their assigned "demands" for each combat period. Demands represent targets and the numbers of missiles required to kill those targets, all of which are assumed known in a given scenario.

Kress, Penn and Polukarov have recently developed an efficient algorithm for solving a similar problem in the context of ground combat. However, their technique does not apply to our problem, because it assumes each combat unit is assigned to meet the demand (for munitions) that arises in its own sector of operation. Our "fully flexible allocation model," called "FFAM," is more appropriate for naval warfare where it is quite possible that all targets will be in range of each combat ship in the fleet, and the force command will determine ship-to-target assignments in the theater of operations. We formulate FFAM as a two-stage stochastic integer program.

FFAM proves difficult to solve. A case involving four ships, five scenarios in the first period and four scenarios in the second period, generates 5,503 equations in 6,494 variables. An optimality gap of more than 10% remains for most instances of this problem after an hour of computation using the CPLEX 7.5 solver on a Pentium IV, 2 GHz personal computer.

To solve problems of practical size, we introduce an operational requirement on acceptable allocation plans. In particular, we require that ships with larger inventories be assigned targets that have higher demands. We believe that a force commander would likely use such an assignment plan in the heat of battle. We also develop lower bounds on single-ship inventories and the total number of missiles put to sea in each combat period. These techniques help reduce solution times by improving the linear-

programming lower bound for the integer program, while only slightly worsening the optimal integer solution, if at all. These modifications add 595 constraints and 53 variables to the model instance described above, resulting in a net reduction of 291 constraints and 890 variables following preprocessing, and the optimal solution time is reached in roughly three minutes. These improvements are limited, however. A instance of the modified FFAM involving six ships and five scenarios in each period, generates 10,305 equations and 14,672 variables. It can be solved in less than an hour only for some cost ratios.

I. INTRODUCTION

Two of the key questions that military logisticians ask are: How many supply items should be procured, and how should the procured supply be distributed to combat units given the uncertainty of combat? This thesis develops models and solution methods to help answer instances of these questions in the context of naval surface warfare.

A. ANTI-SHIP CRUISE MISSILES AND THEIR SUPPLY CHAIN

Anti-ship cruise missiles (ASCMs) constitute the major weapon system of modern navies that do not rely on airpower (for example, the fleets of Denmark, Greece, and the Netherlands [Baker 2002]). As such, a navy requires plans to prescribe the total number of missiles that should be bought, the initial allocation plan of missiles to ships, and, by implication, the number of missiles to be stored in one or more depots, to be distributed at a later time. An optimal plan must identify a minimum-cost package and meet operational requirements with a sufficiently high probability of success. Uncertainty arises because there are a variety of potential combat scenarios, with distinct missile requirements, for which the navy must prepare.

We define a combat scenario, or simply “scenario,” as a set of missions and their associated “demands” with which the combat forces will have to contend in a single period of combat. A mission consists of one or more targets that must be prosecuted, and “demand” denotes the (deterministic) number of missiles that must be fired to successfully prosecute the mission. The duration of a combat period is not specified in days, but rather is the time between successive opportunities to replenish supplies. In the naval context, a combat period can last several days, and is marked by successive port calls. For a fleet to successfully prosecute an entire scenario, we assume that all demands must be met using the ships’ available supply of missiles. (It is possible to relax this assumption and require that only a fixed fraction of the demands be met to successfully prosecute a scenario. However, this constitutes a simple variant of our model we shall not consider.) The planning horizon may include several periods of combat in succession, and the probability of each scenario is conditional on the scenario that

actually occurred in the previous period. We refer to a single set of demands occurring across all time periods as a "compound scenario." This thesis focuses on two-period supply problems with one potential replenishment for each ship.

We assume that a set of scenarios with their respective probabilities has already been defined by planners, and that the numbers of missiles (demands) required to prosecute each mission successfully in each scenario have been established as an extension of these war plans. The scenarios can represent different hypothesized battles as well as uncertain demands within each battle. The duration of the conflict we plan for is usually long enough that combatants have the opportunity to replenish their supplies of missiles, often more than once [Rabinovitch 1997, p. 252]. We assume that missiles cannot be transferred directly from one ship to another, and that no ship will make a port call to offload missiles for the use of some other ship. Thus, any requirement for missiles that a ship might have following the first period of combat can only be met by missiles stored in onshore depots.

A ship's total missile requirement for assigned missions in a compound scenario may exceed its carrying capacity, and the ship will be forced to return to port to load one or more missiles after the first period of battle. But, even if a ship's carrying capacity is sufficient to meet the largest conceivable total demand, it may be preferable not to load the ship to that level. Following a period-I scenario, the probabilities for period-II scenarios are updated. It is possible then that high-demand scenarios no longer seem likely, and the requirements for certain ships can be lowered. We wish to avoid the situation where, after a period of combat, some ships hold excess missiles that may be needed by other units in the next period. Replenishment from central locations is advantageous then, since it utilizes "risk pooling" (e.g., Simchi-Levi *et al.* [2000, p. 116]) to reduce the total number of missiles that must be deployed in the first time period. "Risk pooling" refers to the risk of shortages due to high demands, and not to the risk of losing missiles onboard disabled ships. (It is clear that reducing the number of deployed missiles reduces the number of missiles that can be lost to enemy actions, but we ignore this source of risk in this thesis. We also ignore the risk that arises from the potential interdiction of depot-level inventory.

One of the key features that distinguishes the problem of procuring and deploying military supplies from other supply problems is the singularity of war [Kress 2002, p. 242]. Because we expect the planned supply chain to be tested in war only once, following which new scenarios will need to be developed and planned for, our models will incorporate probability requirements, i.e., "probabilistic constraints," or "chance constraints." (For example, see Birge and Louveaux [1997, pp. 103-108].) This contrasts with the objective functions based on expected values that are often used in civilian supply-chain planning models that apply to repetitive scenarios over relatively long time horizons [Diwekar 2002].

One of the common features with our problem and stochastic supply-chain models in the civilian sector is "recourse." After one period of combat, we can resupply ships in order to improve our chances for success in the second time period. This action constitute "recourse" defined in the stochastic-programming literature (e.g., Birge and Louveaux [1997, pp. 84-100, 122-127]). Thus, the model we eventually create will be a multi-stage stochastic-programming problem with recourse. However, the model is unusual in that a typical multi-stage stochastic program would incorporate uncertainty through expected values and not probabilistic constraints.

We present work that directly extends research initiated by Professor Moshe Kress, Professor Michal Penn, and their student Maria Polukarov [Kress, Penn and Polukarov 2003], hereafter referred to as "KPP." These authors propose a model to optimize the procurement and deployment of supplies to ground combat units in a two-period combat-logistics problem. A set of possible combat scenarios, each with matching demands and an associated probability of occurrence, is defined for each time period. The first combat period is followed by an opportunity to replenish the supplies of all of the units; supplies can be moved between units for purposes of replenishment, and/or a unit may replenish is supplies from a central depot. Optimality in this case implies a minimum-cost procurement and deployment plan such that all of the units meet their required demands in a specified fraction of the scenarios that might arise, weighted by probability of occurrence.

The setting of their model captures the basics of land combat, in which units are assigned different sectors of a battlefront, and must meet the opposition in that sector. We refer to the demands in each scenario as being "rigidly assigned," because once a scenario unfolds, each combat unit is tasked with a specific mission and the demand for munitions induced by that mission.

KPP allow backorders on demands and assume that supplies can be redistributed following the first combat period. Under these conditions, if the cost of allocating supplies to the units is at least as high as the cost of allocating them to a depot, then the two-period problem can be decomposed into two single-period problems and solved sequentially to obtain the two-period optimal solution. KPP develop an exact algorithm to solve the single-time-period problem.

The conditions that lead to the simple solution of the ground-combat problem do not hold for naval combat. A key difference between ground combat and naval surface combat is the flexibility in mission assignment within a specific scenario. A high degree of "interchangeability" arises in meeting demands because ASCM ranges are often long compared with the distances between opposing combat ships when a battle actually takes place [Hughes 1995]. Consequently, several potential ships, or "shooters," can usually "cover" a particular target, and the actual assignment of shooters to targets is decided in real-time. Another difference is that combat ships cannot normally transfer missiles directly among themselves like ground-combat units can; and we will not allow a ship to be called into port to offload missiles and through that action enable indirect-inter-ship transfers of missiles. Finally, the monetary costs of allocating missiles to ships and to the depot are effectively equal, and the cost ratio is used to reflect operational preferences with respect to the location of stocks required for the second period. Thus, we see that new techniques will be required to solve "the KPP problem" in the context of naval combat.

B. THESIS OUTLINE

This thesis extends the models of KPP to suit the setting of naval surface warfare through the use of integer programming. In chapter two, we present a detailed account of

the KPP model for ground combat supply chains. For purposes of clarity and comparison, we show how to state the KPP model as an integer program. In chapter three we essentially extend that integer program to the context of naval combat, with the key difference being that missions (targets) are flexibly assigned to the combat ships. We present solution results for problems of various levels of complexity. In chapter four, we develop operational constraints and valid inequalities that reduce run times and may provide more realistic results.

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II. THE KPP MODEL: GROUND COMBAT

In this chapter we present a detailed account of the two-period supply-chain model proposed by Kress, Penn and Polukarov [Kress et al 2003]. The assumptions of this model suit ground combat, but we will use it as a departure point for an analogous problem in the setting of the naval surface warfare.

A. VERBAL DESCRIPTION

Let n denote the number of combat units on the planner's side. We assume that a set S_{I} of potential scenarios is defined for the first period of combat ("period I"). Each scenario has an associated demand vector, with each element of the vector denoting the demand to be satisfied by the corresponding combat unit. We assume that these demands are for a single, generic commodity, e.g., ammunition, measured in discrete units, e.g., tons measured to the nearest tenths of a ton. There is a one-to-one correspondence between a demand and the unit that can satisfy that demand, and we therefore refer to this scheme as a "rigid assignment of demands." A probability of occurrence $p_s > 0$ is assigned to each scenario $s \in S_{\mathrm{I}}$, such that $\sum_{s \in S_{\mathrm{I}}} p_s = 1$.

For each $s \in S_{\mathrm{I}}$, there is a finite set $S_{\mathrm{II}}(s)$ of possible demand vectors for period II. That is, each period-I scenario has a finite set of period-II scenarios that might follow. Each scenario $s' \in S_{\mathrm{II}}(s)$ has an associated probability $p_{s'|s} > 0$, such that $\sum_{s' \in S_{\mathrm{II}}(s)} p_{s'|s} = 1 \;\; \forall s \in S_{\mathrm{I}}$. The sets $S_{\mathrm{II}}(s)$ may not be disjoint or even different for the various $s \in S_{\mathrm{I}}$.

Each combat unit carries supplies and attempts to satisfy its own demand. Other supplies are stored at a central depot that can ship those supplies to units that require them. Because the lead-time is effectively one period, the units must rely on their initial stocks to satisfy any demands in the first period. Following the first period, shipments of supplies are made to the units that cover any backorders from the first time period, and the forecast demand for the second period. We assume these supplies are munitions in

what follows. Replenishments are made from the central depot or by inter-unit transfers. This problem is an extension of a *two-stage stochastic program with recourse* [Vajda 1972, pp. 27-29; Kolbin 1977, pp. 37-75].

The following provides a verbal description of the KPP model, annotated with constraint numbers that correspond to the mathematical model, which is specified subsequently. This formulation is equivalent to the one proposed by KPP, but, with one exception, replaces the integer variables used in [Kress et al 2003] with binary variables in order to provide a tight linear-programming (LP) relaxation.

- Minimize the cost to procure munitions and allocate some subset of them to n combat units and store the rest as a depot, (2.1).

  Subject to:
- Each period-I scenario is successfully covered only if all the demands in that scenario are satisfied, (2.2).
- Each ship is allocated a specific number of missiles, (2.3).
- The cumulative probability of occurrence of scenarios in which all units meet their period-I demands exceeds a specified threshold, (2.4).
- A unit has a single level of munitions (may be negative) in each period, (2.5), (2.16).
- A scenario is either fully covered or it is not, (2.6), (2.17).
- Each period-II scenario is successfully covered only if any backorder remaining from the preceding period-I scenario and all the demands in the current scenario are satisfied, (2.10).
- For each period-I scenario, the cumulative probability of successfully covered period-II scenarios exceeds a specified threshold, (2.11).
- The total supply distributed to the units between periods of combat does not exceed the amount kept in the depot, (2.12).
- The number of munitions stored in the depot is non-negative, (2.15).

B. MATHEMATICAL DESCRIPTION

The following formulation as a linear integer program sharpens the description.

Indices:

$i \in I$	combat units
$k \in K$	level (number) of munitions

$s \in S_{\mathrm{I}}$ scenario s in period I

$s' \in S_{\mathrm{II}}(s)$ scenario s' in period II

Parameters [units]

p_s probability that period-I scenario s occurs

$p_{s'|s}$ probability that period-II scenario s' occurs, given period-I scenario s occurs

P_{I} probability threshold for period I

P_{II} probability threshold for period II

$d(i,s)$ demand associated with the *i*th unit's mission in scenario *s* [munitions]

c_0 unit cost of procuring a munition and storing it in the depot [$/munition]

c_1 unit cost of procuring a munition and placing it with a combat unit [$/munition]

Decision Variables [units]

x_{ik}^{I} 1 if unit *i* has *k* supplies before the first combat period, and 0 otherwise

x_0 number of munitions allocated initially to the central depot [munitions]

$x_{i,k',k'',s}$ 1 if unit *i* has exactly k' munitions left after scenario s, and k'' munitions are added at the replenishment opportunity, and 0 otherwise

x_{iks}^{II} 1 if unit *i* is replenished to level *k* munitions following period-I scenario *s*, and 0 otherwise

y_{iks} 1 if unit *i* receives *k* munitions following period-I scenario *s*, and 0 otherwise

z_s 1 if the demand vector in period-I scenario s is met by the allocation plan, and 0 otherwise

$z_{s'|s}$ 1 if the demand vector in period-II scenario s' is met by the allocation plan given that scenario *s* occurs in period I, and 0 otherwise

Formulation

$$\min_{\mathbf{x,y,z}} c_0 x_0 + c_1 \sum_i \sum_k k x_{ik}^{\mathrm{I}} \tag{2.1}$$

s.t.

Period I:

$$z_s \le \sum_{k \ge d(i,s)} x^{\mathrm{I}}_{ik} \quad \forall i \in I,\, s \in S_{\mathrm{I}} \tag{2.2}$$

$$\sum_k x^{\mathrm{I}}_{ik} = 1 \quad \forall i \in I \tag{2.3}$$

$$\sum_{s \in S_{\mathrm{I}}} p_s z_s \ge P_{\mathrm{I}} \tag{2.4}$$

$$x^{\mathrm{I}}_{ik} \in \{0,1\} \quad \forall i \in I,\, k \in K \tag{2.5}$$

$$z_s \in \{0,1\} \quad \forall s \in S_{\mathrm{I}} \tag{2.6}$$

Period II:

$$x_{i,k'-d(i,s),k'',s} \le x^{\mathrm{I}}_{ik'} \quad \forall i \in I,\, k',\, k'' \in K,\, s \in S_{\mathrm{I}} \tag{2.7}$$

$$x_{i,k',k'',s} \le y_{ik''s} \quad \forall i \in I,\, k',\, k'' \in K,\, s \in S_{\mathrm{I}} \tag{2.8}$$

$$x^{\mathrm{II}}_{iks} = \sum_{k'} \sum_{k'' \mid k'+k''=k} x_{i,k',k'',s} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{2.9}$$

$$z_{s'|s} \le \sum_{k \ge d(i,s')} x^{\mathrm{II}}_{iks} \quad \forall i \in I,\, s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{2.10}$$

$$\sum_{s' \in S_{\mathrm{II}}(s)} p_{s'|s} z_{s'|s} \ge P_{\mathrm{II}} \quad \forall s \in S_{\mathrm{I}} \tag{2.11}$$

$$x_0 - \sum_i \sum_k k y_{iks} \ge 0 \quad \forall s \in S_{\mathrm{I}} \tag{2.12}$$

$$x_{i,k',k'',s} \in \{0,1\} \quad \forall i \in I,\, k' \in K,\, k'' \in K,\, s \in S_{\mathrm{I}} \tag{2.13}$$

$$x^{\mathrm{II}}_{iks} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{2.14}$$

$$x_0 \in \mathbb{Z}^+ \tag{2.15}$$

$$y_{iks} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{2.16}$$

$$z_{s'|s} \in \{0,1\} \quad \forall s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{2.17}$$

The inventory remaining for unit i following the first period of combat, $k - d(i,s)$, may be negative for scenarios that require more munitions than the unit was allocated (the scenarios' cumulative probability does not exceed $1 - P_{\mathrm{I}}$.) KPP assume that each unit maintains a safety stock of supplies, and if the required demand is higher than planned, the safety stock is tapped to satisfy the remaining demand. If a scenario's demand is met by dipping into any unit's safety stock, then that scenario is

"unsuccessful." KPP further assumes that the safety stock is replenished along with the regular inventory for every period-II scenario that unit is expected to cover. The assumption of a safety stock allows backorder of supplies. If that assumption is restricted, they replace constraints (2.10) with

$$x_{i,(k'-d(i,s))_+,k'',s} \le x^{\mathrm{I}}_{ik'} \quad \forall i \in I,\, k',\, k'' \in K,\, s \in S_{\mathrm{I}}, \tag{2.18}$$

where $k_+ \equiv \max\{k,0\}$.

KPP assume that the cost associated with storing supplies at the depot is no higher than the cost associated with allocating them to the units, i.e., $c_1 \ge c_0$. This assumption is reasonable in ground combat for two reasons: Increasing the amount of supplies allocated to a combat unit increases the effort needed to transport them as the unit moves and protect them close to the front; furthermore, increasing the logistics attachment to a combat unit increases the operational burden on the commander.

Under the conditions mentioned above (possible back-order, possible transshipment during resupply and $c_1 \ge c_0$), the two-period model can be decomposed into two separate one-period problems and solved sequentially. In the first problem, we minimize the number of supplies allocated to the units in order to satisfy demands in the first combat period with a specified probability. The numbers of supplies allocated to the combat units for the first period of combat, $\hat{x}_{ik}$, $i = 1,...,n$ for all *k*, are used as parameters in the second-period problem, where we minimize the number of supplies to be added to the units' inventories from the depots. Each of those problems is a special case of a combinatorial problem that KPP calls the *Minmax Subset Problem* (MSP). Solving the two MSP problems sequentially yields an optimal two-period allocation. KPP provide an efficient exact algorithm to solve the MSP [Kress et al 2003].

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III. NAVAL COMBAT – FLEXIBLE ASSIGNMENTS

The KPP model lacks two important features of naval combat. More important is the flexible assignment of missions to ships. That is, ships are not automatically assigned to the target (mission) in "their sector" as ground combat units are. Rather, they can be assigned to many different targets, perhaps any target, and the force commander makes those assignments after the targets have presented themselves. Furthermore, the assumption of a safety stock is unreasonable for combat ships, which typically have onboard inventories of at most eight ASCMs [Baker 2002]. This chapter formulates an integer-programming model that allows full flexibility in target assignment, i.e., any ship may prosecute any target, and each ship may use only its current missile supply. This model extends to include some assignment restrictions by removing variables that represent prohibited assignments.

A. NAVAL COMBAT MODELS

We wish to obtain optimal initial ship load-outs and depot inventory level to ensure that the probability of all ships successfully meeting their demands exceeds specified thresholds for each combat period. We assume two-periods of combat, between which a ship may return to port to replenish its supply. We do not allow ships to be recalled to port to offload missiles for use by other ships, nor is direct inter-ship transfer of missiles allowed. Consequently, no inter-ship transfer of missiles is allowed whatsoever. The number of missiles to be loaded on each ship in the second period is a recourse variable, which is decided upon after the first period of combat is completed.

In naval surface warfare, the planning of allocations is complicated by the fact that once a set of targets is evident, the combat forces have significant freedom in assigning targets to shooters. Targets are assigned among the available shooters based on available inventory and tactical positions. In some cases, each target is within range of all potential shooters, and we refer to this situation as "fully flexible." However, the tactical situation does not normally allow every ship to engage any target, and we refer to the targets in each scenario as being "semi-flexibly assigned." The extension of a fully

flexible model to a semi-flexible one is readily achieved by eliminating appropriate assignment variables, and we present only the fully flexible model.

In the land-combat setting of KPP, we assume that each unit maintains sufficient safety stock with which to carry out any mission even when their initial supply cannot. (Safety stock must be replenished, however, if it is used.) Safety stock does not make sense in a naval setting and we assume none exists. We further assume the existence of lower and upper bounds on the number of missiles that will be carried aboard a ship because of capacity constraints, operational considerations and doctrine. We assume that if the required number of missiles to cover the assigned mission exceeds the number available onboard the ship, the ship will "stay and fight" and expend its entire inventory. We consider such scenarios to be unsuccessful. A small adjustment can be made to model a situation where a shooter prefers not to engage a target if the shooter is insufficiently armed.

In the naval case, the monetary costs of loading the missiles on the ships prior to the first combat period or storing them at shore-based facilities are equivalent because a ship's nominal capacity is prepaid in the construction process. Increasing that capacity requires prohibitive changes in hardware and software, and will not be considered. Because of the cost equivalence, the ratio c_0 / c_1 is used to convey operational preferences regarding allocation strategy. If we wish to reduce the operational burden of carrying out wartime replenishment operations in port, we set $c_0 / c_1 > 1$. If we wish to avoid the risk of losing missiles due to own-force casualties or expenditure on low-quality targets, we can set $c_0 / c_1 < 1$.

The ratio's value determines the relative value we assign to missiles initially allocated to ships compared with those stored for period II. Essentially, whenever the ratio is not one, we are willing to buy more than the minimum amount required in order to obtain an allocation plan that suits us. For example, setting $c_0 / c_1 = 1.25$ means that we view a solution that procures five missiles and allocates them to the ships, to be equally desirable as a solution that requires us to procure only four missiles that must be stored at the depot.

B. TWO-PERIOD, FULLY FLEXIBLE ASSIGNMENT MODEL

We develop here the Fully Flexible Assignment Model (FFAM) to minimize the procurement cost of missiles required to satisfy demands in projected scenarios, while satisfying a user-specified minimum probability of success. A scenario is considered satisfied if all ships have enough missiles to cover their assigned missions in that scenario. In the first time period, the model procures a set of missiles and allocates a specific number of those missiles to each ship. And, it assigns one mission to each ship in each scenario. We assume that the number of missions is no greater than the number of ships; if there are fewer missions than ships, the data can be modified so that the model assigns some ships to artificial missions with zero demand. Following the first period, the model calculates each ship's remaining inventory and supplements that inventory as necessary. Missions in period-II scenarios are assigned so that period-II scenarios are successfully covered with a user-specified probability that can depend on the preceding period-I scenario. The model also ensures that each ship's inventory is maintained within required limits at the outset of each combat period.

Because mission assignments are fully flexible, two scenarios in the same time period are essentially identical if one scenario's demand vector is a permutation of the other's. For the sake of solution efficiency, if identical scenarios are presented as input data, they should be consolidated and appropriate adjustments made to the probability data. (Or, if an automatic scenario generator is used to create scenarios, it should be adjusted to avoid producing identical scenarios.)

The following provides a verbal description of FFAM, annotated with constraint numbers that correspond to the mathematical model, which is specified subsequently. Data are also specified for a version of the model in which the total initial inventory is fixed, i.e., has already been procured.

1. FFAM Verbal Description

- Minimize the cost to procure a set of missiles and allocate some subset of those missiles to $|I|$ combat ships and store the remainder at a depot (3.1).

  Subject to:

- A scenario is successfully covered only if every ship has enough missiles to satisfy the demand of its assigned mission, (3.2), (3.22).
- Each ship is allocated a specific number of missiles, (3.3).
- One mission is assigned to each ship in each scenario, (3.4), (3.25).
- Exactly one ship (with a specific number of missiles) is assigned to each mission in each scenario, (3.5), (3.26).
- In each period, the probability of successfully covering the scenarios exceeds a user-specified threshold, (3.6), (3.24). In the second period, the cumulative probability must be achieved for every possible, preceding, period-I scenario.
- Following assignment and prosecution of a mission, each ship maintains an inventory equal to its initial level minus the demand associated with its assigned mission, if that demand can be met, (3.7). Otherwise, the remaining inventory is zero, (3.8).
- A ship's post-scenario inventory equals the number of missiles remaining in its inventory after prosecuting mission *m*, (3.9).
- There is exactly one level of missiles remaining on a ship, (3.10).
- The number of missiles allocated to ship *i*+1 does not exceed the number allocated to ship *i*, (3.11). (These constraints breaks some of the symmetries in the problem, thereby accelerating the branch-and-bound procedure, and require that the upper bounds on missile capacities be listed in non-increasing order.)
- The number of missiles that may be placed on each ship is bounded from below (operational constraint) and from above (physical capacity limit), (3.12), (3.27).
- A ship's total inventory of missiles, before prosecuting any period-II mission, equals the post-mission inventory after period I plus any missiles that are replenished, (3.18)-(3.20).
- Only one level of replenishment may take place after a period-I scenario, (3.21).
- The total number of missiles distributed to the units between periods of combat may not exceed the amount kept in storage, (3.23).

2. FFAM Mathematical Description

Indices

$i \in I$	ships
$k \in K$	level (number) of missiles
$m \in M$	missions
$s \in S_{\mathrm{I}}$	scenario s in period I
$s' \in S_{\mathrm{II}}(s)$	scenario s' in period II, following scenario s

Parameters [units]

p_s	probability that period-I scenario *s* occurs
$p_{s'\|s}$	probability that period-II scenario s' occurs, given period-I scenario *s* occurs
P_{I}	probability threshold for period I
P_{II}^{s}	probability threshold for period II, if scenario *s* occurred in period I
$d(m,s)$	demand associated with mission *m* in scenario *s* [missiles]
c_1	unit cost of procuring a missile and allocating it to a ship [$/missile]
c_2	unit cost of procuring a missile and storing it in the depot [$/missile]
lb_i	lower bound on the number of missiles that may be allocated to ship *i* [missiles]
ub_i	upper bound on the number of missiles that may be allocated to ship *i* [missiles]
b_{tot}	total procured inventory (used in the fixed-inventory version) [missiles]

Decision Variables [units]

x_{ik}^{I}	1 if ship *i* has *k* missiles before the first combat period, and 0 otherwise
x_{II}	number of missiles allocated initially to the central depot [missiles]
r'_{ikms}	1 if ship *i* has *k* missiles remaining following mission *m* in period-I scenario *s*, and 0 otherwise
r_{iks}	1 if ship *i* has *k* missiles left following period-I scenario *s*, and 0 otherwise
y_{iks}	1 if ship *i* receives *k* missiles following period-I scenario *s*, and 0 otherwise
$x_{i,k',k'',s}$	1 if ship *i* has exactly k' missiles left after scenario *s*, and k'' missiles are added at the replenishment opportunity, and 0 otherwise
x_{iks}^{II}	1 if ship *i* is replenished to level *k* missiles following period-I scenario *s*, and 0 otherwise
z_s	1 if the demand vector in period-I scenario *s* is met by the allocation plan, and 0 otherwise

$z_{s'|s}$ 1 if the demand vector in period-II scenario s' is met by the allocation plan given that scenario s occurs in period I, and 0 otherwise

u^s_{ikm} 1 if ship i has k missiles and is assigned mission m in period-I scenario s, and 0 otherwise

$u^{s'|s}_{ikm}$ 1 if ship i has k missiles and is assigned mission m in period-II scenario s', following scenario s in period I, and 0 otherwise

Formulation

$$\min_{\mathbf{u,x,y,z}} c_1 \sum_i \sum_k k x^{\mathrm{I}}_{ik} + c_2 x_{\mathrm{II}} \tag{3.1}$$

s.t.

Period I:

$$z_s \le \sum_m \sum_{k \ge d(m,s)} u^s_{ikm} \quad \forall i \in I,\, s \in S_{\mathrm{I}} \tag{3.2}$$

$$\sum_k x^{\mathrm{I}}_{ik} = 1 \quad \forall i \in I \tag{3.3}$$

$$\sum_m u^s_{ikm} = x^{\mathrm{I}}_{ik} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{3.4}$$

$$\sum_i \sum_k u^s_{ikm} = 1 \quad \forall m \in M,\, s \in S_{\mathrm{I}} \tag{3.5}$$

$$\sum_{s \in S_{\mathrm{I}}} p_s z_s \ge P_{\mathrm{I}} \tag{3.6}$$

$$u^s_{i,k+d(m,s),m} = r^+_{ikms} \quad k \ge 1,\, \forall i \in I,\, m \in M,\, s \in S_{\mathrm{I}} \tag{3.7}$$

$$\sum_{k' \le d(m,s)} u^s_{i,k',m} = r^+_{ikms} \quad k = 0,\, \forall i \in I,\, m \in M,\, s \in S_{\mathrm{I}} \tag{3.8}$$

$$\sum_m r'_{ikms} = r_{iks} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{3.9}$$

$$\sum_k r_{iks} = 1 \quad \forall i \in I,\, s \in S_{\mathrm{I}} \tag{3.10}$$

$$\sum_{k' \ge k} x^{\mathrm{I}}_{ik'} \ge \sum_{k' \ge k} x^{\mathrm{I}}_{i+1,k'} \quad \forall i \in I - \{n\},\, k \in K \tag{3.11}$$

$$x^{\mathrm{I}}_{ik} \equiv 0 \quad \forall i,\, k \,|\, (k < lb_i) \cup (k > ub_i) \tag{3.12}$$

$$r'_{ikms} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, m \in M,\, s \in S_{\mathrm{I}} \tag{3.13}$$

$$r_{iks} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{3.14}$$

$$x_{ik}^{\mathrm{I}} \in \{0,1\} \quad \forall i \in I,\, k \in K \tag{3.15}$$

$$z_s \in \{0,1\} \quad \forall s \in S_{\mathrm{I}} \tag{3.16}$$

$$u_{ikm}^{s} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, m \in M,\, s \in S_{\mathrm{I}} \tag{3.17}$$

Period II:

$$x_{i,k',k'',s} \le r_{ik's} \quad \forall i \in I,\, k,\, k' \in K,\, s \in S_{\mathrm{I}} \tag{3.18}$$

$$x_{i,k',k'',s} \le y_{ik''s} \quad \forall i \in I,\, k,\, k' \in K,\, s \in S_{\mathrm{I}} \tag{3.19}$$

$$x_{iks}^{\mathrm{II}} = \sum_{k'} \sum_{k''|k'+k''=k} x_{i,k',k'',s} \quad \forall i \in I,\; k \in K,\; s \in S_{\mathrm{I}} \tag{3.20}$$

$$\sum_{k} y_{iks} = 1 \quad \forall i \in I,\, s \in S_{\mathrm{I}} \tag{3.21}$$

$$z_{s'|s} \le \sum_{m} \sum_{k \ge d(m,s')} u_{ikm}^{s'|s} \quad \forall i \in I,\, s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{3.22}$$

$$\sum_{i} \sum_{k} k\, y_{iks} \le x_{\mathrm{II}} \quad \forall s \in S_{\mathrm{I}} \tag{3.23}$$

$$\sum_{s' \in S_{\mathrm{II}}(s)} p_{s'|s} z_{s'|s} \ge P_{\mathrm{II}}^{\mathrm{s}} \quad \forall s \in S_{\mathrm{I}} \tag{3.24}$$

$$\sum_{m} u_{ikm}^{s'|s} = x_{iks}^{\mathrm{II}} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{3.25}$$

$$\sum_{i} \sum_{k} u_{ikm}^{s'|s} = 1 \quad \forall m \in M,\, s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{3.26}$$

$$x_{iks}^{\mathrm{II}} \equiv 0 \quad \forall i,\, k \,|\, (k < lb_i) \cup (k > ub_i),\, s \in S_{\mathrm{I}} \tag{3.27}$$

$$y_{iks} \in \{0,1\} \quad \forall i \in I,\; k \in K,\, s \in S_{\mathrm{I}} \tag{3.28}$$

$$x_{i,k',k'',s} \in \{0,1\} \quad \forall i \in I,\, k' \in K,\, k'' \in K,\, s \in S_{\mathrm{I}} \tag{3.29}$$

$$x_{iks}^{\mathrm{II}} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, s \in S_{\mathrm{I}} \tag{3.30}$$

$$x_{\mathrm{II}} \in \mathbb{Z}^{+} \tag{3.31}$$

$$z_{s'|s} \in \{0,1\} \quad \forall s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{3.32}$$

$$u_{ikm}^{s'|s} \in \{0,1\} \quad \forall i \in I,\, k \in K,\, m \in M,\, s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{3.33}$$

This model contains some variables that can be substituted out, namely x_{ik}^{I}, r_{iks}, and x_{iks}^{II}. Defining these variables in constraints (3.4), (3.9), and (3.20) generates "branching constraints," and branching on these variables may accelerate the branch-and-bound solution for the integer model; see Appleget and Wood [2000].

By restricting the acceptable assignments, this model can be easily adapted to a setting similar to that of the KPP model. We fix each mission to its corresponding unit by setting

$$u_{ikm}^{s} = 0 \quad \forall i \in I,\, k \in K,\, m \neq i,\, s \in S_{\mathrm{I}} \tag{3.34}$$

and

$$u_{ikm}^{s'|s} = 0 \quad \forall i \in I,\, k \in K,\, m \neq i,\, s \in S_{\mathrm{I}},\, s' \in S_{\mathrm{II}}(s) \tag{3.35}$$

and relying on constraints (3.5) and (3.26) to assign mission m to ship i. This formulation, called "FFAM-kpp," solves for any cost ratio, and does not allow safety stocks or transshipment of supplies. To allow transshipment, a modification allowing negative levels of missiles is required.

If the total missile inventory has already been set, we seek a missile allocation plan to maximize the probability that the ships will have sufficient inventory to satisfy the ensuing demands in both time periods. We replace the objective function (3.1) with

$$\max_{\mathbf{u},\mathbf{x},\mathbf{y},\mathbf{z}} \sum_{s \in S_{\mathrm{I}}} p_s z_s + \sum_{s \in S_{\mathrm{I}}} \sum_{s' \in S_{\mathrm{II}}(s)} p_s p_{s'|s} z_{s'|s} \tag{3.36}$$

and replace constraints (3.6) and (3.24) by the constraint

$$x_{\mathrm{II}} + \sum_{i} \sum_{k} k x_{ik}^{\mathrm{I}} \leq b_{tot}\,. \tag{3.37}$$

Note that, in this formulation, we assign equal importance to satisfying demands in the first and second periods.

This objective offers an alternative for solving the procurement problem. We can successively reduce b_{tot} and find the minimum for which the probability of success is acceptable. Because this objective function expresses average probabilities across periods and among scenarios, this is not equivalent to the formulation of FFAM, and we do not pursue it further in this thesis.

C. RESULTS

We wish to ascertain the applicability of the FFAM model to problems of practical size. We believe that the smallest real-world problems will have three or four ships, with perhaps five scenarios defined in each period. A large, realistically-sized problem might have as many as ten ships, and perhaps ten scenarios defined for each combat period. These scenarios will not, in general, be fully flexible, because of the geographical dispersion of ships an analyst is likely to hypothesize. However, we examine the computational behavior of only fully-flexible models because preliminary computational experience indicates that these are the most difficult to solve. That is, the solutions times we report may be viewed as conservative estimates of the effort required to solve corresponding semi-flexible (restricted) models.

We define two cases to test the behavior of FFAM. In each case we define the number of ships, and the sets of period-I and period-II scenarios with their respective probabilities and demands. These parameters control the feasible region of the problem. For ease of presentation, we let $S_{\mathrm{II}} \equiv \bigcup_{s \in S_{\mathrm{I}}} S_{\mathrm{II}}(s)$, and set $p_{s'|s} = 0$ if $s' \notin S_{\mathrm{II}}(s)$.

We solve all of the models in this thesis using the CPLEX solver version 7.5 [ILOG 2003] on a Pentium IV, 2 GHz personal computer with 1 Gbyte of RAM. We seek the optimal solution, and generally set the relative optimality criteria to 0.0%. In cases involving integer costs, an absolute gap of 0.99 is used.

1. Test Case 1

In this case, we assume there are three ships available, and that $S_{\mathrm{I}} = \{s_1, s_2, s_3\}$ and $S_{\mathrm{II}} = \{s'_4, s'_5\}$. The parameters given in Table 1 complete the definition of this case. We must be able to satisfy the demands arising in any two of the three period-I scenarios, and subsequently in both period-II scenarios. Each ship must carry between two and eight missiles into combat during each period. Notice that we must allocate more than eleven missiles, the highest aggregate demand, to the ships in period I in order to be able to satisfy any two scenarios. Therefore, we can expect some ships to carry missiles over into the second period regardless of which period-I scenario occurs and which allocation plan is chosen.

p_s	$\frac{1}{3}\ \forall s$
$p_{s'\|s}$	$\begin{array}{cccc} & s_1 & s_2 & s_3 \\ s_4 & 0.5 & 0.3 & 0.7 \\ s_5 & 0.5 & 0.7 & 0.3 \end{array}$
P_{I}	0.66
$P_{\mathrm{II}}^{\mathrm{s}}$	$1\ \forall s$
$d(m,s)$ $s \in S_{\mathrm{I}}$ $s' \in S'_{\mathrm{I}}$	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s'_4 & s'_5 \\ m_1 & 5 & 1 & 6 & 4 & 3 \\ m_2 & 3 & 5 & 2 & 5 & 2 \\ m_3 & 3 & 5 & 2 & 4 & 7 \end{array}$
lb_i	$2\ \forall i$
ub_i	$8\ \forall i$

Table 1. Parameter Specifications for Case 1.

The table lists the probabilities of each scenario (p_s, $p_{s'|s}$), the success thresholds specified for each period (P_{I} , $P_{\mathrm{II}}^{\mathrm{s}}$), the demands associated with the various missions in each scenario$\left(d(m,s)\right)$, and the capacity limits imposed on each ship (lb_i, ub_i).

2. Test Case 2

In this case there are four combat ships, and $S_{\mathrm{I}} = \{s_1, \ldots, s_5\}$ and $S_{\mathrm{II}} = \{s'_6, \ldots, s'_9\}$. The parameters given in Table 2 complete the definition of this case. Given the probability requirements, we must be able to satisfy the demands arising in scenario s_1 plus any three of the remaining four scenarios in period I. And, in period II, we must be able to satisfy the demands in scenario s'_6 plus any two of the remaining three scenarios.

p_s	$\left(\frac{2}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)$
$p_{s'\|s}$	$(0.4,0.2,0.2,0.2)\ \forall s\in\{s_1,\ldots,s_5\}$
P_{I}	0.83
$P_{\mathrm{II}}^{\mathrm{s}}$	$0.8\ \forall s$
$d(m,s)$ $s\in S_{\mathrm{I}}$	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ m_1 & 3 & 4 & 4 & 3 & 4 \\ m_2 & 0 & 3 & 5 & 5 & 1 \\ m_3 & 3 & 2 & 4 & 4 & 4 \\ m_4 & 3 & 1 & 2 & 5 & 5 \end{array}$
$d(m,s')$ $s'\in S_{\mathrm{II}}$	$\begin{array}{ccccc} & s'_6 & s'_7 & s'_8 & s'_9 \\ m_1 & 3 & 4 & 4 & 3 \\ m_2 & 0 & 3 & 5 & 5 \\ m_3 & 3 & 2 & 4 & 4 \\ m_4 & 3 & 1 & 2 & 5 \end{array}$
lb_i	$2\ \forall i$
ub_i	$8\ \forall i$

Table 2. Parameter Specifications for Case 2.
All labels defined as in table 1.

3. Computational Results

We refer to a specific cost ratio applied to a case as a "subcase." We solve five subcases for each of the two cases, varying the cost of storing missiles in the depot, c_2, while holding the cost of allocating missiles to ships constant, i.e., $c_1 = 1$. Naturally, varying the cost ratio changes the objective function and results in different optimal solutions, but the computational effort required to solve the model instances changes as well. Note that for a fixed value of c_2 (and c_1), all optimal solutions with the same value of $x_{\mathrm{I}} \equiv \sum_i \sum_k k x_{ik}^{\mathrm{I}}$, the number of missiles assigned to the ships in the first period, have

the same objective value. In much of the following discussion, we refer to x_{I}^* as the first-period solution and to the integer pair $\mathbf{x}^* = \left(x_{\mathrm{I}}^*, x_{\mathrm{II}}^*\right)$ as the solution of the full problem.

Table 3 summarizes the results for test case 1. For each value of c_2, we specify the optimal solution $\mathbf{x}^*$, the number of missiles initially allocated to each ship in that solution, the number of nodes in the branch-and-bound tree required to solve the model, and the computation time required. Case 1 generates 2,207 equations in 1,982 variables.

In the case that $c_1 = c_2 = 1$, we are only interested in minimizing the total number of missiles required for the planning horizon. Any feasible allocation requiring 26 missiles is optimal, and we can see that the solver happens upon one such combination. When there is a preference for allocating missiles to the ships or to maintaining them in the depot, we are reduced to a single optimal solution in this test case. There can, however, be several optimal solutions, depending on the scenario data and the cost ratio. Note that as c_2 / c_1 deviates away from 1.0, the optimal solution requires less computational effort to identify.

This case exemplifies the value of maintaining some missiles in reserve, even when the cost of doing so is double the cost of allocating those missiles to ships. The solution for the subcase with $c_2 = 2$ implies that we can allocate 23 missiles in the first period, and require three additional missiles to be replenished from the depot. The ships' capacities allow us to initially allocate 24 missiles, eight to each ship. If we do that, the ships' inventories following each of the scenarios s_1, s_2 and s_3 are (5,5,3), (7,3,3), and (6,6,2), respectively. In order to satisfy both period-II scenarios, we must replenish the ships' inventories up to (7,4,4) missiles. Because missiles cannot be moved from ship to ship, if scenario s_1 occurs, we supplement the ship with five missiles by two and the ship with three missiles by one. Similarly, if scenario s_3 occurs we also require three missiles to be added to the ships' inventories. Thus, any solution allocating 24 missiles to the ships in period I does not reduce the number of missiles required at the depot (compared with initially allocating 23 missiles), and cannot be optimal.

c_2	Solution $\mathbf{x}^* = \left(x_{\mathrm{I}}^*, x_{\mathrm{II}}^*\right)$	Initial Allocation	# Nodes	Time [sec]
0.5	(12,14)	(6,3,3)	97	10
0.9	(12,14)	(6,3,3)	1088	44
1.0	(15,11)	(7,4,4)	1308	51
1.1	(23,3)	(8,8,7)	1212	42
2.0	(23,3)	(8,8,7)	391	20

Table 3. Solution Results for Case 1.

The "Solution" column gives the optimal number of missiles to be loaded for period-I combat and the number of missiles to be stored for later use. "Initial Allocation" gives initial load-outs for the ships. The columns labeled "# Nodes" and "Time" indicate the number of nodes in the branch-and-bound tree and the time required to reach a solution, respectively.

The results for the second case are displayed in table 4. This case generates 5,503 equations in 6,543 variables, which are reduced by the CPLEX preprocessor to 2797 equations in 4056 variables. Possibly because of a high level of symmetry, this model is very difficult to solve optimally. We report the best integer solution found, and the relative optimality gap [GAMS 1998] after one hour of computation. The incumbent integer solution when $c_2 = 0.9$ is not optimal, since the solution (15,15) is better.

The solution for the subcase with $c_2 = 2$ is also not optimal. Suppose we had chosen to load all the ships to capacity in the first period. Then we would have been left with sufficient onboard inventories to meet the period-II probability threshold following every period-I scenario except s_4: The inventory remaining after s_4 is (5,4,3,3), and neither scenario s_9' nor s_{10}' can be satisfied. But, replenishing the third ship with a single missile suffices to satisfy the demands of scenario s_9', and ensures that the threshold requirement, $P_{\mathrm{II}}(s_4) = 0.8$, is satisfied. Thus, the solution (32,1) is also feasible, and costs 34 units, two less than (28,3), the best solution found. Solving case 2 in FFAM-kpp by incorporating (3.35) into the model requires three seconds or less for each subcase.

c_2	Solution	Initial Allocation	Relative Optimality Gap
0.5	(15,15)	(5,4,4,2)	4.8%
0.9	(17,13)	(5,4,4,4)	12.2%
1.0	(15,15)	(5,4,4,2)	13.3%
1.1	(19,11)	(7,4,4,4)	14.5%
2.0	(28,3)	(8,8,8,4)	11.8%

Table 4. Solution Results for Case 2.

The "Solution" column gives the number of missiles to be loaded for period-I combat and the number of missiles to be stored for later use, for the best integer solution found. The "Initial Allocation" gives the initial load-outs for the ships. The "Relative Optimality Gap" column gives the relative difference between the objective of the quoted solution and the global lower bound at termination of the branch-and-bound procedure.

C. CONCLUSIONS

This chapter has developed FFAM (Fully Flexible Assignment Model) to identify an optimal missile-procurement and allocation plan for two periods of naval surface combat, with one replenishment opportunity. We assume that any ship can engage any target, but exactly one ship will do so. We further assume that missiles loaded onto ships will not be unloaded, even if we wish to redistribute them.

The model encounters computational difficulty even when solving problems of modest size. In a problem with four ships and nine scenarios, the model fails to find, in one hour of computation time on a 2 GHz personal computer, an answer guaranteed to be within 10% of optimality for most problem instances. The next chapter develops methods to improve the model's run-time performance.

IV. IMPROVING MODEL PERFORMANCE

We wish to improve FFAM's computational performance significantly, because realistic problem instances may have significantly more than the four ships and nine scenarios that are already difficult to solve. We do this by adding constraints in various ways to tighten the model's LP relaxation. One set of constraints restricts target assignment plans to reflect operational directives. Other constraints tighten the LP relaxation by enforcing lower bounds on the number of missiles that each ship requires individually, and that the combat fleet requires as a whole, to meet the demands of each combat period. These lower bounds are found by solving instances of single-period inventory models. The constraints we propose in this chapter are tailored to the fully-flexible model.

A. TIGHTENING THE FFAM FORMULATION

1. Operational Constraints

In a number of the problems examined in the previous chapter (for example, case 2, subcases $c_2 = 1$ and $c_2 = 0.5$), the solver finds an optimal integer solution fairly quickly. The solver cannot, however, declare the solution to be optimal in a reasonable amount of time (taken arbitrarily to be one hour) because the LP lower bound is weak. That is, too many nodes in the branch-and-bound tree must be explored to bring up the weak lower bound. We substantially tighten the lower bound by restricting the problem, yet only increase the optimal solution value slightly.

Finding an optimal assignment is difficult for the model, and obviously very difficult for a force commander to determine in real-time. We expect the commander to follow some assignment heuristic when faced with multiple targets posing varying demands. A plausible assignment heuristic would require that ships with larger inventories be assigned targets with higher associated demands. In particular, the result of such a heuristic in scenarios where the demand cannot be fully met is that the plan *minimizes the maximum deficiency in single-target assignments*. A proof of this fact follows the mathematical description of the constraints.

This assignment heuristic contrasts with the FFAM model, in which an assignment plan may attempt to spare missiles for the next time period if scenario demands cannot be fully met. It can do this by assigning ships with low inventories to targets with high demand. Of course, the new assignment heuristic restricts the problem and may require an optimal plan to procure more missiles than in the unconstrained model, but that allocation plan would enable the commander to find a "reasonable" assignment plan in every scenario.

As an example, consider a scenario with three targets and the corresponding demand vector (4,4,2), and suppose the ships have available the inventories (4,3,1). Under the proposed heuristic, the *i*th ship is assigned to the *i*th target. If the commander were to switch the last two assignments, the second ship would not use its entire inventory, but the second target would be attacked with only a single missile.

We implement this assignment restriction for both assignment periods. We first reorder the demands in each scenario in non-increasing order, so that

$$d(m,s) \geq d(m+1,s) \quad m = 1,\ldots,|M|-1,\ \forall s \in S_{\mathrm{I}} \cup S_{\mathrm{II}}. \tag{4.1}$$

Since the ships are initially allocated missiles in non-increasing order—recall constraints (3.11)—we simply assign the missions to the ships in that order for the first period, obtaining the constraints

$$\sum_{k} u_{ikm}^{s} = 1 \quad \forall i \in I,\ k \in K,\ m = i,\ s \in S_{\mathrm{I}}. \tag{4.2}$$

We enforce these assignment restrictions by implementing constraints (3.34), used in the FFAM-kpp formulation. The actual assignments are different from those obtained by FFAM-kpp, since the ships first reorder the targets by demand levels.

Ship inventories in period II are not necessarily ranked sequentially, so more complex constraints are required to ensure that any mission m is assigned to a ship with at least as many missiles as are allocated to the ship assigned to mission $m+1$:

$$\sum_{i}\sum_{k' \geq k} u_{ik'm}^{s'|s} \geq \sum_{i}\sum_{k' \geq k} u_{ik',m+1}^{s'|s} \quad \forall k \in K,\ m = 1,\ldots,|M|-1,\ s \in S_{\mathrm{I}},\ s' \in S_{\mathrm{II}}(s). \tag{4.3}$$

To prove that the plan minimizes the maximum deficiency in single-target assignments, observe that constraints (3.11) imply that in every period-I scenario, we

have $q_1 \geq q_2 \geq \cdots \geq q_{|I|}$, where $q_i = \sum_k kx_{ik}$ is the number of missiles allocated to ship i in period I. Constraints (4.2) imply that we assign shooter i to demand i. Let $i'(s)$ denote a ship index where the maximum deficiency, $\delta_{i'(s)} = \max_i \left\{ d(i'(s), s) - q_{i'(s)}, 0 \right\}$, occurs for scenario s. Consider a different assignment plan in scenario s, where the ship with $q_{i'(s)}$ missiles is assigned a mission with demand $d(i''(s), s)$ and the ship with $x_{i''(s)}$ missiles is assigned a mission with demand $d(i'(s), s)$. If $i''(s) > i'(s)$, then $d(i'(s), s) - q_{i''(s)} \geq \delta_{i'(s)}$; conversely, if $i''(s) < i'(s)$, then $d(i''(s), s) - q_{i'(s)} \geq \delta_{i'(s)}$. The period-II missile allocations are not ordered, by index, but constraints (4.3) guarantee that the inventory of ship assigned to mission m is no less than that of the ship assigned to mission $m+1$. Therefore, the logic of proof for period-I scenarios still applies. Hence, the proposed heuristic minimizes the maximum deficiency in every scenario.

2. Valid Inequalities

Here we develop valid inequalities (integer cutting planes) based on the solution of single-period problems. Define v_1 as the minimum aggregate ship load-out required to satisfy the period-I scenarios and $v_2(s)$ as the minimum aggregate ship load-out required to satisfy the period-II scenarios following scenario s in period I. We can and do obtain v_1 by solving a single-period model, defined by constraints (3.1)-(3.17) and (4.2), *sans* the explicit calculation of the remaining inventory (constraints (3.7)-(3.10) and (3.13)), but where we set $c_1 = 1$ and $c_2 = 0$. We refer to this model as "FFAM-sp." We obtain $v_2(s)$ by solving FFAM-sp on the period-II scenarios $s' \in S_{\mathrm{II}}(s)$ with their respective demands, probabilities and specified probability thresholds. Hence, FFAM-sp is solved $|S_{\mathrm{I}}|$ times, yielding a value $v_2(s)$ each time.

We can use v_1 and $v_2(s)$ to generate lower bounds on the number of missiles that must be allocated to combat ships in each period. If more missiles are required to satisfy the demands for a respective combat period than the ships can carry in any of the single-period problems, that problem is infeasible, and we can avoid solving the full, two-period model. Because any feasible solution to the two-period problem must satisfy the first-

period constraints, we obtain $x_{\mathrm{I}} \geq v_1$. Initial testing indicates that this is best expressed by adding the following constraints, and explicitly branch on q_i:

$$q_i = \sum_k k x_{ik}^{\mathrm{I}} \quad \forall i \tag{4.4}$$

$$\sum_i q_i \geq v_1. \tag{4.5}$$

Similarly, the total number of missiles allocated to the ships in the second period must reach at least $v_2(s)$ following each period-I scenario, leading to these valid inequalities:

$$\sum_i \sum_k k x_{iks}^{\mathrm{II}} \geq v_2(s) \quad \forall s \in S_{\mathrm{I}} \tag{4.6}$$

3. Single-Ship Lower Bounds

The lower bound on a single ship's inventory, set in constraints (3.12) and (3.27), is derived from generic operational constraints. Tighter constraints can be derived from specific problem data.

We define a modified single-period problem called "FFAM-lb" (for "lower bound") in which we require the set of $|M||S_{\mathrm{I}}|$ missions to be assigned to the ships as before. Constraints (3.2)-(3.6), (3.11) and (4.2) comprise this model, together with the integrality requirement. We define z_{is} to equal 1 if ship *i* can satisfy the demand required by its assigned mission in period-I scenario *s*, and 0 otherwise, and replace z_s by z_{is} in constraints (3.2) and (3.6), so that each ship selects its set of successful scenarios independently of the other ships (although mission assignments are not independent.) This relaxes the single-period problem because each ship can reach the specified probability threshold P_{I} by satisfying demands from a set of scenarios different from those of another ship.

We solve FFAM-lb and obtain an optimal allocation plan $\hat{\mathbf{q}} = \left\{\hat{q}_1, \cdots, \hat{q}_{|I|}\right\}$ that defines the number of missiles *k* allocated to each ship. We then modify the lower bounds on the capacities by setting

$$x_{ik}^{\mathrm{I}} \equiv 0 \quad \forall i,\, k \mid k < \hat{q}_i \tag{4.7}$$

before solving FFAM.

These inequalities are valid because constraints (4.2) completely control the period-I target assignment plans. Therefore, the target-to-shooter assignment plans generated by FFAM-lb are identical to those generated by FFAM when constraints (4.2) are applied. The decisions actually left to FFAM in period I and to FFAM-lb are how to allocate a sufficient number of missiles so that the probability threshold will be met. This is equivalent to choosing a subset of "successful" scenarios with sufficient weight.

Formally, we show that the allocation plan obtained by solving FFAM-lb, denoted here by $\underline{\mathbf{q}}^* = \left\{ \underline{q}_1^*, \cdots, \underline{q}_{|I|}^* \right\}$, is (a) unique, and (b) maintains $\underline{q}_i^* \leq q_i^* \ \forall i \in I$ compared with any optimal solution $\mathbf{q}^*$ to FFAM. Using simplified notation, FFAM-lb and the period-I portion of FFAM can be stated by:

FFAM (first period)	FFAM-lb
$\min \sum_i q_i$	$\min \sum_i \underline{q}_i$
s.t. $q_i - d(i,s) z_s \geq 0 \quad \forall i \in I, s \in S$	s.t. $\underline{q}_i - d(i,s) z_{is} \geq 0 \quad \forall i \in I, s \in S$
$\sum_{s \in S} z_s p_s \geq P$	$\sum_{s \in S} z_{is} p_s \geq P \quad \forall i \in I$
$lb_i \leq q_i \leq ub_i$	$lb_i \leq q_i \leq ub_i$
$q_i \in Z^+$, $z_s \in \{0,1\}$	$\underline{q}_i \in Z^+$, $z_{is} \in \{0,1\}$

To prove (a), suppose that $\underline{\mathbf{q}}^1$ and $\underline{\mathbf{q}}^2$ are two distinct optimal solutions to FFAM-lb. Then there is some $i' \in I$ for which $\underline{q}_{i'}^1 < \underline{q}_{i'}^2$. Let $S_i^j \subset S$ denote the index set of scenarios that are satisfied by ship i in the jth optimal solution of FFAM-lb. Therefore, $\underline{q}_i^j \geq d(i,s) \ \forall i \in I, s \in S_i^j$, and because we minimize $\sum_i \underline{q}_i$, it follows that $\underline{q}_i^j = \max_{s \in S_i^j} d(i,s) \ \forall i \in I$. Because the index set $S_{i'}^1$ satisfies the constraint $\sum_{s \in S_{i'}^1} z_{is} p_s \geq P$, the solution $\underline{\mathbf{q}}^{2*}$ where $\underline{q}_{i'}^{2*} = \underline{q}_{i'}^1$ and $\underline{q}_i^{2*} = \underline{q}_i^2 \ i \neq i'$ is feasible, in contradiction to the

optimality assumption of $\underline{\mathbf{q}}^2$. The optimal solution is unique, and we denote it by $\underline{\mathbf{q}}^*$ and its corresponding index set by S_i^*.

To prove (b), let $S^* \subset S$ denote the index set of scenarios that are satisfied by an optimal solution $\mathbf{q}^*$ of FFAM. Therefore, $q_i^* = \max_{s \in S^*} d(i,s)\ \forall i \in I$. Now suppose that for some $i' \in I$, $q_{i'}^* < \underline{q}_{i'}^*$. Then we must have that $\max_{s \in S^*} d(i',s) < \max_{s \in S_i^*} d(i',s)$. Since the index set S^* satisfies $\sum_{s \in S^*} z_s p_s \geq P$, we may construct a feasible solution to FFAM-lb, $\underline{\mathbf{q}}^{**}$, where $\underline{q}_{i'}^{**} = q_{i'}^*$ and $\underline{q}_i^{**} = \underline{q}_i^*\ i \neq i'$, thus reducing the objective value by $\left(\underline{q}_{i'}^* - q_{i'}^*\right) > 0$. This contradicts the optimality assumption of $\underline{\mathbf{q}}^*$, and establishes the validity of the lower bounds obtained by FFAM-lb.

Similar logic can be used to tighten the lower bounds on single-ship inventories for the second period. For each period-I scenario s, we set $P_{\mathrm{I}} = P_{\mathrm{II}}(s)$, $p_{s'} = p_{s'|s}$ and solve FFAM-lb on the data pertaining to the period-II scenarios $s' \in S_{\mathrm{II}}(s)$. Because FFAM-lb is a single-period model, we use the notation $\hat{x}_{ik}(s)$ and $\hat{\mathbf{q}}(s)$ to denote the values obtained using period-II data following s. We obtain a minimal allocation plan $\hat{\mathbf{q}}(s)$ and the associated values $\hat{x}_{ik}(s)$. The allocation plan states the number of missiles loaded on each ship prior to the second combat period. Since FFAM-lb contains constraints (3.11), its solution guarantees that $\hat{q}_1 \geq \hat{q}_2 \geq \cdots \geq \hat{q}_{|I|}$.

Because some ships may retain missiles following period I, it is not known, at this point, which ship will actually be loaded to each level required by the plan. Hence, the period-II assignment plan is also unknown prior to solving FFAM. We may choose, for example, to allocate the highest inventory to ship $i = |I|$ rather than to ship $i = 1$. Constraints (4.3), however, ensure that for a given set of ship inventory levels, the target assignment plan is unique (to within permutations of missions with equal demands). Therefore, the allocation plan $\hat{\mathbf{q}}(s)$ is a permutation of the single-ship inventory lower bounds (derived for ship inventories conforming to $\hat{q}_1 \geq \hat{q}_2 \geq \cdots \geq \hat{q}_{|I|}$). Let us define the

parameter n_{ks} as the number of ships that require at least *k* missiles in period II following period-I scenario *s*, and let

$$n_{ks} = \sum_i \sum_{k' \geq k} \hat{x}_{ik'}(s). \tag{4.8}$$

This leads to the following valid inequality, which we add to FFAM:

$$\sum_i \sum_{k' \geq k} x^{\mathrm{II}}_{ik's} \geq n_{ks} \quad \forall s \in S_{\mathrm{I}}. \tag{4.9}$$

4. Summary of Procedure

The modified FFAM model, "FFAM-m" hereafter, consists of all the constraints defined in chapter 3 plus constraints (4.2)-(4.6), (4.7) and (4.9). We reorder the mission requirements in non-increasing order of demand, as specified by inequality (4.1). In order to obtain the parameters required by these constraints, we solve $|S_{\mathrm{I}}|+1$ instances of FFAM-sp and $|S_{\mathrm{I}}|+1$ instances of FFAM-lb. These single-period models solve very quickly compared to FFAM-m, so we ignore this computational effort when describing results.

B. RESULTS FOR FFAM-m

We repeat both cases defined in chapter 3 (case 1 and case 2) to examine performance and solution trade-offs. We then subject the model to a different test case, case 3, in which the probabilities of period-II scenarios and the required threshold are actually different following each period-I scenario. Finally, we increase the problem size to establish the practical limits of the model (case 4.)

1. Test Case 1

Here we reorder the demands of each scenario in non-increasing order, in accordance with (4.1), and run the necessary cases of FFAM-sp and FFAM-lb to obtain the parameters required for the valid inequalities in FFAM-m (constraints (4.4)-(4.6), (4.7) and (4.9)). We list those parameters at the top of table 5, followed by the results presented in the same manner as in chapter 3.

$(v_1; \mathbf{v}_2(s))$	$(12; (15, 15, 15))$			
$\hat{\mathbf{q}}$	(5,3,2)			
n_{ks}	$\begin{array}{cccc} & s_1 & s_2 & s_3 \\ k_4 & 3 & 3 & 3 \\ k_7 & 1 & 1 & 1 \end{array}$			
c_2	Solution $\mathbf{x}^* = (x_{\mathrm{I}}^*, x_{\mathrm{II}}^*)$	Initial Allocation	# Nodes	Time [Sec]
0.5	(12,14)	(6,3,3)	21	2.5
0.9	(12,14)	(6,3,3)	106	4.5
1.0	(14,12)	(6,4,4)	93	4.5
1.1	(23,3)	(8,8,7)	115	4.5
2.0	(23,3)	(8,8,7)	45	3.4

Table 5. FFAM-m Solution Results for Case 1.

The solutions obtained have the same objective value as those obtained by FFAM. The solution time is reduced by 85.7% on average, when compared with the same instances solved using FFAM (table 3). $(v_1; \mathbf{v}_2(s))$ gives the lower bounds on the number of missiles required by the ships to satisfy demands in periods I and II, respectively. $\hat{\mathbf{q}}$ is the vector of lower bounds on period-I inventories for the ships. n_{ks} specifies the number of ships that require at least *k* missiles following period-I scenario *s*. The rest of the table entries are defined as in table 3.

FFAM-m generates 2,349 equations in 1,985 variables, an increase of 142 equations and three variables compared with FFAM. By comparing the performance of FFAM-m to that achieved by FFAM (see table 3), we conclude that the number of nodes in the branch-and-bound solution drops by 88.1%, on average. The time to reach the solution is reduced by 85.7%, on average. The solution for each cost ratio is the same as those derived by FFAM, except for the subcase with $c_1 = c_2$, where a different allocation plan is chosen with the same objective value.

A major factor in the observed solution-time reductions must be the decrease in the initial integrality gap which the branch-and-bound algorithm must overcome. Table 6 lists the integer objective values Z_{opt}, the values of the LP relaxations Z_{LP}, and the relative "integrality gaps," defined as $\left(Z_{opt} - Z_{LP}\right)/Z_{LP}$, for both FFAM and FFAM-m. The relative integrality gap is reduced by 45.5% on average for FFAM-m. The optimal objective values for the two models are the same for all instances, but may differ, in general.

		FFAM		FFAM-m		
c_0	Z_{opt}	Z_{LP}	Integrality Gap	Z_{LP}	Integrality Gap	Improvement
0.5	19.0	11.62	63.5%	14.72	29.1%	54.2%
0.9	24.6	12.58	95.5%	15.97	54.0%	43.5%
1.0	26.0	12.79	103.2%	16.26	60.3%	41.6%
1.1	26.3	12.98	102.6%	16.54	59.0%	42.5%
2.0	29.0	14.74	96.7%	18.99	52.7%	45.5%

Table 6. Comparisons of Integrality Gaps, Case 1.

Z_{opt} gives the optimal objective value for the subcase defined by c_0. Z_{LP} and "Integrality Gap" state the solution of the LP relaxation for that subcase and the resulting gap that the branch-and-bound procedure must overcome. The "improvement" column states the relative decrease in gap size between the two models.

2. Test Case 2

The run-time improvement in case 2 (defined in table 2) is more significant than in case 1. FFAM-m generates 6,098 equations in 6,547 variables, an increase of 595 equations and 53 variables compared to FFAM. Preprocessing reduces the problem size to 2506 equations in 3166 variables, a net reduction of 192 equations and 890 variables compared with FFAM. Before modifying FFAM, the integer solutions found could not

be proven, in one hour of computation, to be within 10% of optimality for most cost ratios tested. Following modification, the optimal answer is achieved in roughly three minutes, except for $c_2 = 0.5$, where it is achieved in 35 seconds. Results for case 2 are listed in table 7.

$(v_1; \mathbf{v}_2(s))$	$(15;\ (15,\ 15,\ 15,\ 15,\ 15))$			
$\hat{\mathbf{q}}$	(5,4,4,2)			
n_{ks}	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ k_2 & 4 & 4 & 4 & 4 & 4 \\ k_3 & 3 & 3 & 3 & 3 & 3 \\ k_4 & 2 & 2 & 2 & 2 & 2 \end{array}$			
c_2	Solution $\mathbf{x}^* = (x_{\mathrm{I}}^*, x_{\mathrm{II}}^*)$	Allocation	# Nodes	Time [Sec]
0.5	(15,15)	(5,4,4,2)	45	35
0.9	(15,15)	(5,4,4,2)	245	145
1.0	(18,12)	(8,4,4,2)	284	176
1.1	(18,12)	(8,4,4,2)	324	152
1.15	(26,5)	(8,8,8,2)	548	189
2.0	(32,1)	(8,8,8,8)	638	176

Table 7. FFAM-m Solution Results for Case 2.

FFAM-m solves most instances of the problem in roughly three minutes, whereas the FFAM did not reach a solution guaranteed to be within 10% of optimal in an hour of computation. As we increase c_0, the number of missiles we are willing to procure increases, so that the number put to sea can increase significantly. The table entries are defined as in table 5.

Notice that if we wish to minimize the number of missiles that must be replenished, we require three more missiles in total compared with the optimal solution

that occurs when the costs of missiles for both periods are equal. An intermediate solution requiring 31 missiles in total was also found for the subcase $c_2 = 1.15$. Small changes in c_2 may result in slight differences in the total number procured, but can produce significantly different allocation plans.

Table 8 shows the differences in integrality gap between FFAM and FFAM-m. The improvement is only calculated for four of the subcases because the optimal solution for the case with $c_2 = 1.1$ is not known for FFAM. The other objective values are the same for the two models. The relative gap is reduced by 37.6% on average.

		FFAM		FFAM-m		
c_2	Z_{opt}	Z_{LP}	Gap	Z_{LP}	Gap	Improvement
0.5	22.5	14.96	50.4%	17.81	26.3%	47.8%
0.9	28.5	15.80	80.4%	18.79	51.7%	35.7%
1.0	30.0	15.99	87.6%	18.98	58.1%	33.7%
2.0	34.0	17.36	95.8%	20.73	64.0%	33.2%

Table 8. Comparisons of Integrality Gaps, Case 2.
Table entries are defined as in table 6.

3. Test Case 3

As is clear from the parameter specifications and the values of $\mathbf{v}_2(s)$ and n_{ks} for case 2, there is a high degree of symmetry in requirements following the first period. We construct an alternative case here, case 3, to verify that solution quality does not change significantly when conditional probabilities and probability thresholds differ following each period-I scenario. There are four ships in this case, and $S_{\mathrm{I}} = \{s_1, \ldots, s_5\}$ and $S_{\mathrm{II}} = \{s_6', \ldots, s_9'\}$. The parameters given in Table 9 complete the definition of this case.

p_s	$(0.3, 0.1, 0.2, 0.2, 0.2)$
$p_{s'\mid s}$	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_6 & .4 & .2 & .2 & .3 & .2 \\ s_7 & .2 & .4 & .2 & .2 & .2 \\ s_8 & .2 & .2 & .4 & .2 & .5 \\ s_9 & .2 & .2 & .2 & .3 & .1 \end{array}$
P_{I}	0.8
P_{II}^{s}	$(0.9, 0.8, 0.8, 0.6, 0.7)$
$d(m,s)$ $s \in S_{\mathrm{I}}$	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ m_1 & 6 & 4 & 4 & 7 & 3 \\ m_2 & 3 & 3 & 4 & 5 & 2 \\ m_3 & 3 & 3 & 4 & 4 & 1 \\ m_4 & 0 & 1 & 4 & 2 & 1 \end{array}$
$d(m,s')$ $s' \in S_{\mathrm{II}}$	$\begin{array}{ccccc} & s'_6 & s'_7 & s'_8 & s'_9 \\ m_1 & 5 & 4 & 5 & 5 \\ m_2 & 3 & 4 & 5 & 3 \\ m_3 & 2 & 3 & 4 & 3 \\ m_4 & 0 & 1 & 2 & 3 \end{array}$
lb_i	$2 \ \forall i$
ub_i	$8 \ \forall i$

Table 9. Parameter Specifications for Case 3.
All labels defined as in table 1.

The results for case 3 are listed in table 10. The model's size is the same as for case 2, and the run-time performance is comparable. This problem also produces some interesting results. Since $x_{\mathrm{I}}^* = v_1$ for $c_2 = 0.9$, we are guaranteed that no other solution $\mathbf{x}^*$ exists for any $c_2 \le 0.9$. Similarly, from the solution for subcase $c_2 = 2.0$, we deduce that no feasible solution with $x_{\mathrm{II}}^* = 1$ exists. Although $v_2(s)$ varies by three over the period-I scenarios, the optimal number of missiles that must be procured varies by just

one across the entire range of cost ratios. Comparing the subcases with $c_2 = 0.9$ and $c_2 = 1.1$, which appear to reflect a minor shift in the allocation strategy, we see that 32 missiles are procured in each solution, but the optimal allocation plans are completely different; and the difference is greater than the differences seen for the same costs changes when applied to case 2 (see table 7).

$(v_1; \mathbf{v}_2(s))$	$(18;\ (17, 15, 16, 14, 16))$			
$\hat{\mathbf{q}}$	(6,4,4,2)			
n_{ks}	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ k_2 & 4 & 4 & 4 & 4 & 4 \\ k_3 & 4 & 3 & 3 & 3 & 3 \\ k_4 & 3 & 2 & 3 & 1 & 3 \\ k_5 & 2 & 0 & 2 & 0 & 2 \end{array}$			
c_2	$\mathbf{x}^* = (x_{\mathrm{I}}^*, x_{\mathrm{II}}^*)$	Initial Allocation	# Nodes	Time [Sec]
0.9	(18,14)	(6,4,4,4)	221	120
1.0	(29,3)	(8,8,7,6)	321	147
1.1	(29,3)	(8,8,8,5)	517	158
2.0	(31,2)	(8,8,8,7)	332	127

Table 10. FFAM-m Solution Results for Case 3.

The number of missiles required for period-II combat, $v_2(s)$, varies by three across period-I scenarios, but assignment plans exist such that the total number of missiles prescribed by FFAM-m varies by only one across all possible cost ratios. Table entries are defined as in table 5.

4. Test Case 4

Naturally, as the numbers of ships and scenarios increase, the computational effort required to solve FFAM-m increases. But, the actual effort in a particular case may also vary significantly with the cost ratio, demands, and scenario probabilities. A

comprehensive analysis of model behavior is beyond the scope of this thesis, but we can obtain some intuition about FFAM-m's limitations in the following case, case 4.

We first tested FFAM-m, increasing the number of ships and scenarios, until a "borderline difficult" ship and scenario instance was found: This instance involves six combat ships, and $S_{\mathrm{I}} = \{s_1, \ldots, s_5\}$ and $S_{\mathrm{II}} = \{s'_6, \ldots, s'_{10}\}$. For some cost-ratio subcases, optimal solutions are identified relatively quickly, but in other subcases the optimal solution is not found in an hour of computation. The parameters given in Table 11 complete the definition of this case.

Table 12 lists results for case 4, which generates 10,305 equations in 14,672 variables. The obvious observation is that finding the optimal solution when a clear allocation preference exists is much easier than otherwise. In the subcase with $c_2 = 0.9$, nearly an hour of computation time is required to identify an optimal solution with 47 missiles. Optimal solutions to the other two subcases with $c_2 \approx 1$ cannot be identified in one hour of computation. It is easy to verify that the solution (25,22) is better than the incumbent in the subcase with $c_2 = 1$, and that the solution (46,2) is better than the incumbent in the subcase with $c_2 = 1.1$.

B. CONCLUSIONS

The chapter has developed valid inequalities to reduce the time required to find an optimal solution. Coupled with operational constraints based on target-assignment decision-rules that a force commander might use, FFAM's run-time performance has been improved by a factor of at least 20 for larger problems. Some practical problems are still difficult to solve, however.

p_s	$0.2\ \forall s \in S_{\mathrm{I}}$
$p_{s'\mid s}$	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_6 & .2 & .2 & .2 & .25 & .25 \\ s_7 & .2 & .2 & .2 & .25 & .25 \\ s_8 & .2 & .2 & .2 & .25 & .25 \\ s_9 & .2 & .2 & .2 & .25 & 0 \\ s_{10} & .2 & .2 & .2 & 0 & .25 \end{array}$
P_{I}	0.8
$P_{\mathrm{II}}^{\mathrm{s}}$	$0.8\ s \in \{s_1, s_2, s_3\}$ $1\quad s \in \{s_4, s_5\}$
$d(m,s)$ $s \in S_{\mathrm{I}}$	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ m_1 & 6 & 7 & 4 & 7 & 4 \\ m_2 & 4 & 4 & 4 & 5 & 3 \\ m_3 & 3 & 3 & 4 & 4 & 1 \\ m_4 & 2 & 3 & 4 & 3 & 1 \\ m_5 & 2 & 2 & 4 & 3 & 1 \\ m_6 & 0 & 1 & 2 & 1 & 1 \end{array}$
$d(m,s')$ $s' \in S_{\mathrm{II}}$	$\begin{array}{cccccc} & s'_6 & s'_7 & s'_8 & s'_9 & s'_{10} \\ m_1 & 6 & 5 & 5 & 6 & 7 \\ m_2 & 5 & 5 & 3 & 4 & 5 \\ m_3 & 2 & 4 & 3 & 4 & 4 \\ m_4 & 1 & 4 & 3 & 3 & 3 \\ m_5 & 1 & 3 & 3 & 3 & 1 \\ m_6 & 1 & 2 & 3 & 1 & 0 \end{array}$
lb_i	$2\ \forall i$
ub_i	$8\ \forall i$

Table 11. Parameter Specifications for Case 4.

Parameters are defined as in table 1.

$(v_1, \mathbf{v}_2(s))$	(24; 25, 25, 25, 25, 26)			
$\hat{\mathbf{q}}$	(7, 4, 4, 3, 3, 2)			
n_{ks}	$\begin{array}{cccccc} & s_1 & s_2 & s_3 & s_4 & s_5 \\ k_2 & 6 & 6 & 6 & 6 & 6 \\ k_3 & 5 & 5 & 5 & 6 & 6 \\ k_4 & 3 & 3 & 3 & 4 & 4 \\ k_5 & 1 & 1 & 1 & 2 & 2 \\ k_6 & 0 & 0 & 0 & 1 & 1 \\ k_7 & 0 & 0 & 0 & 0 & 1 \end{array}$			
c_2	$\mathbf{x}^* = (x_{\mathrm{I}}^*, x_{\mathrm{II}}^*)$	Initial Allocation	Abs. Gap	Time [Sec]
0.4	(24,24)	(7,5,4,3,3,2)	0	661
0.9	(25,22)	(7,4,4,4,4,2)	0	3307
1.0	(28,20)	(8,7,4,3,3,3)	1.00	3600*
1.1	(27,21)	(8,4,4,4,4,3)	2.65	3600*
2.2	(46,2)	(8,8,8,8,7,7)	0	888

Table 12. FFAM-m Solution Results for Case 4.

The optimal solution is difficult to find when $c_2 \approx c_1$. The "Abs. Gap" column gives the difference between the best integer solution found and the global lower bound when the solution procedure terminates. Note that Abs. Gap can actually be rounded down to the nearest integer only for integer costs. The asterisks indicate subcases that were terminated following one hour of computation. The rest of the table entries are defined as in table 5.

V. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

This thesis has extended the two-period ground-combat inventory model of "KPP," i.e., Kress, Penn and Polukarov [2003], to the setting of naval surface warfare. Those authors propose a model to optimize the procurement and deployment of supplies to ground combat units in a two-period combat-logistics problem. A set of possible combat scenarios, each with matching demands and an associated probability of occurrence, is defined for each time period. The first combat period is followed by an opportunity to replenish the supplies of all of the units, as needed. Optimality in this case implies a minimum-cost procurement and deployment plan such that all of the units meet their required demands in a specified fraction of the scenarios that might arise, weighted by probability of occurrence.

We explore two main extensions of the KPP model. First, we enable targets to be flexibly assigned to the combat ships (units), after the force commander identifies the target set. In contrast, KPP assumes that a scenario defines each unit's specific mission. We assume that target assignments are "fully flexible" and one-to-one, meaning that each ship can be assigned any mission, and each mission is assigned to only one ship. (Dummy targets can be created if the number of ships exceeds the number of targets.) The second extension of the KPP model eliminates the assumption of an indeterminate safety stock, which KPP uses to satisfy mission requirements when the allocated supplies, in the first time period, are insufficient. Our models explicitly account for every missile in the inventory, and recognize the limited missile-carrying (inventory) capacity of each ship.

We model the two-period supply problem as a stochastic integer-programming model, which we call "FFAM." This program determines plans that allocate missiles to ships and assign ships to missions (targets) in order to satisfy the demands associated with those missions with the specified probability. Obtaining the optimal solution for this program can be difficult, however. A modest-sized problem (case 2), involving four ships, five period-I scenarios and four period-II scenarios, generates 5,503 equations in

6,494 variables, which are reduced by the CPLEX preprocessor to 2797 equations in 4056 variables. An optimality gap of more than 10% remains for most instances of this problem after an hour of computation by CPLEX on a Pentium IV, 2 GHz computer.

To solve problems of practical size, we introduce an operational requirement on acceptable allocation plans that requires ships with larger inventories be assigned to targets having higher demands; we call the modified model "FFAM-m." We believe that a force commander would likely use such an assignment plan in the heat of battle. We also develop lower bounds on single-ship inventories and the total number of missiles put to sea in each combat period. These serve to reduce the (processed) problem size by 192 equations and 890 variables and improve the linear-programming lower bound, thereby reducing solution times by at least 95% (in case 2).

For large problem instances that are especially hard to solve, we develop methods to find optimal solutions for the special cases in which we are operationally inclined to favor minimizing the number of missiles that must be replenished or the number of missiles put to sea in the first period. These methods appear to provide good approximations of the optimal solution for problems in which the inclination is unclear. We relax some of the integrality constraints to obtain a lower bound on the optimal objective value. The application of these methods appears to yield good solutions quickly. For a problem involving six ships, eight period-I scenarios and five period-I scenarios (generating 16,452 equations and 23,438 variables), an upper and a lower bound, yielding an optimal solution, are found in about 20 minutes when minimizing the total number of missiles to be procured. Attempting to solve the model directly yields a gap of four missiles after an hour of computation.

B. FUTURE WORK

Flexible target assignments will always arise naval combat, but in most situations, and in particular as force sizes grow, the assignments are not fully flexible. Hard restrictions stemming from such issues as missile ranges can easily be added to the models to reflect geographically impractical assignments. The validity of the techniques used to modify FFAM needs to be established for semi-flexible settings. If the semi-flexible model can be implemented simply by eliminating certain variables in the FFAM-

m formulation, solution times will likely be reduced. In any case, advanced solution techniques that solve larger problem instances in practical time should be developed.

Realistic target assignments are not necessarily restricted to be one-to-one. Targets may be assigned to more than one ship in some situations, and the required demand can be shared. This may be a realistic option, but it may also be undesirable from an operational point of view; penalties with varying weights could be added to discourage this option's use, as appropriate. Similarly, a ship may prosecute more than one target, particularly if the number of targets exceeds the number of ships, and our models should be extended to allow this possibility.

A third issue that merits attention arises from the fact that assigning targets in the real world is a stochastic process. In practice, not all of the targets in a scenario are detected at the time an attack begins, and some shooters will have to be committed to targets before the force commander can see the full extent of the attack. Consequently, targets may be identified or at least assessed incorrectly, causing the expenditure of too many or too few missiles. A special model or sub-model will be required to handle such situations (for example, see Washburn [2001]). A practical allocation plan should be robust to the effects of combat uncertainty, and techniques to ensure this should be explored.

Finally, we note the need for handling this issue: potential enemy interdiction of our assets. Interdiction may occur through (a) direct attack on the missile depots, or (b) disruption of access to those depots by submarines or offensive mining operations. Furthermore, missiles that are placed on ships may be lost in combat if those ships come under attack. In theory it is not too difficult to add scenarios to incorporate potential interdictions, but actually solving such a model might require substantial research effort.

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LIST OF REFERENCES

Appleget, J.A. and Wood, R.K., (2000). "Explicit-Constraint Branching for Solving Mixed-Integer Programs," *Computing Tools for Modeling, Optimization and Simulation*, eds. Laguna, M. and Gonzalez-Velarde, J.L., Kluwer Academic Publishers, Boston, pp. 243-261.

Baker III, A. D., (2002). *The Naval Institute Guide to Combat Fleets of the World*, U.S. Naval Institute Press, Annapolis MD.

Birge, J. and Louveaux, F., (1997). *Introduction to Stochastic Programming*, Springer-Verlag, New York.

Diwekar, U. M., (2002). *Optimization under Uncertainty: an Overview*, SIAG-OPT Views-and-News, 13, no. 1, pp. 1-8.

GAMS Development Corporation (1998) *GAMS: A User's Guide*, Washington, DC.

Hughes, W. P. Jr., (1995). "A Salvo Model of Warships in Missiles Combat Used to Evaluate Their Staying Power," *Warfare Modeling*, eds. Bracken, J., Kress, M. and Rosenthal, R. E., John Wiley and Sons, Inc., Danvers, MA, pp 121-144.

ILOG CPLEX. [http://www.ilog.com]. December 10, 2003.

Kolbin, V. V., (1977). *Stochastic Programming*, D. Reidel Publishing Company, Dordrecht, Holland.

Kress, M., (2002). *Operational Logistics – The Art and Science of Sustaining Military Operations*, Kluwer Academic Publishers, Boston, MA.

Kress, M., Penn, M., and Polukarov, M., (2003). "Two-Stage Supply-Chain with Recourse and Probabilistic Constraints," in review.

Polukarov, M., (2002). "A Two-Period Supply Problem with Stochastic Demands," MS in Operations Research, Department of Industrial Engineering and Management, Technion, Israel.

Rabinovitch, A., (1997). *The Boats of Cherbourg*, Naval Institute Press, Annapolis, MD.

Simchi-Levi, D., Kaminsky, P., and Simchi-Levi, E., (2000). *Designing and Managing the Supply Chain*, Irwin McGraw-Hill, Boston.

Vajda, S., (1972). *Probabilistic Programming*, Academic Press, New York.

Washburn, A., (2001). *Joint Optimizing Informational Strike Tool*, NPS-OR-02-001-PR Project Report, Naval Postgraduate School, Monterey, California.

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